Year 13 Maths

Using the substitution $u = \sin^2(x)$, or otherwise, find

$$\int \sqrt{\frac{1}{x} - 1}\, dx$$

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Let $x = \sin^2(u)$

Differentiating:

$$\frac{dx}{du} = 2\sin(u)\cos(u)$$

$$\implies dx = 2\sin(u)\cos(u)\ du$$

Note:

$$\sqrt{\frac{1}{x} - 1}$$

$$= \sqrt{\frac{1}{\sin^2(u)} - 1}$$

$$= \sqrt{\text{cosec}^2(u) - 1}$$

$$= \sqrt{\cot^2(u)}$$

$$= \cot(u)$$

Now to the integral:

$$\begin{aligned}\int \sqrt{\frac{1}{x} - 1}\, dx \\ = \int \cot(u) \times 2\sin(u)\cos(u)\, du \\ = \int 2\frac{\cos(u)}{\sin(u)} \times \sin(u)\cos(u)\, du \\ = \int 2\cos^2(u)\, du \\ = \int 1 + \cos(2u)\, du \\ = u + \frac{1}{2}\sin(2u)\end{aligned}$$

Now to substitute x back in:

$$\begin{aligned} x = \sin^2(u) \\ \implies \sin(u) = \sqrt{x} \\ \implies u = \sin^{-1}(\sqrt{x})\end{aligned}$$

Note:

$$\frac{1}{2}\sin(2u) = \sin(u)\cos(u) = \sin(u)\sqrt{1 - \sin^2(u)}$$

Therefore:

$$\int \sqrt{\frac{1}{x} - 1}\, dx = \sin^{-1}(\sqrt{x}) + \sqrt{x}\sqrt{1 - x} + c$$