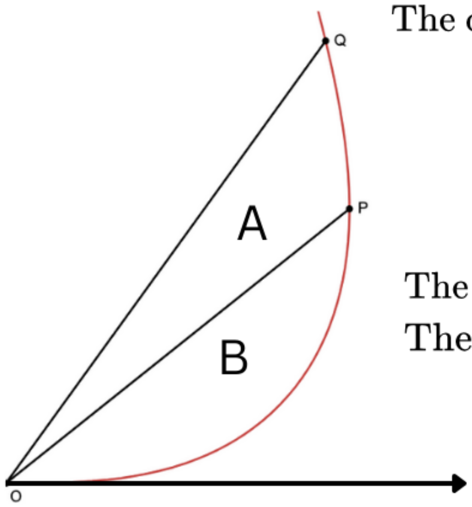
Year 13 Further Maths



The diagram shows the curve with polar equation

$$r = \sqrt{\tan\theta},\ 0 \leq \theta < \frac{\pi}{2}$$

The point $P(p, \alpha)$ has tangent perpendicular to the initial line

The point Q has polar coordinates (q, β)

The area of A is equal to the area of B

Find the value of β.

mathsrules.co.uk

The point P has tangent perpendicular to the initial line. This means that at point P,

$$\frac{dx}{d\theta} = 0$$

Let's find this derivative:

$$x = rcos(\theta)$$

$$\implies \frac{dx}{d\theta} = r \times -\sin\theta + cos\theta \times \frac{dr}{d\theta}$$

$$r = (\tan\theta)^{\frac{1}{2}} \implies \frac{dr}{d\theta} = \frac{1}{2} \times (\tan\theta)^{-\frac{1}{2}} \times \sec^2\theta$$

So

$$\frac{dx}{d\theta} = \sqrt{\tan\theta} \times -\sin\theta + cos\theta \times \frac{1}{2} \times \frac{1}{\sqrt{\tan\theta}} \times \frac{1}{\cos^2\theta}$$

$$\implies \frac{dx}{d\theta} = -\sqrt{\tan\theta}\sin\theta + \frac{1}{2\sqrt{\tan\theta}cos\theta}$$

$$\implies \frac{dx}{d\theta} = -\frac{2\tan\theta\sin\theta\cos\theta}{2\sqrt{\tan\theta}\cos\theta} + \frac{1}{2\sqrt{\tan\theta}cos\theta}$$

$$\implies \frac{dx}{d\theta} = \frac{-2\tan\theta\sin\theta\cos\theta + 1}{2\sqrt{\tan\theta}cos\theta}$$

$$\implies \frac{dx}{d\theta} = \frac{-2\sin^2\theta + 1}{2\sqrt{\tan\theta}cos\theta}$$

This derivative = 0 therefore:

$$\frac{-2\sin^2\theta + 1}{2\sqrt{\tan\theta}cos\theta} = 0$$

$$\implies -2\sin^2\theta + 1 = 0$$

$$\implies \sin^2\theta = \frac{1}{2}$$

$$\implies \sin\theta = \pm\frac{1}{\sqrt{2}}$$

$\implies \theta = \frac{\pi}{4}$, as θ is positive.

The area of a polar curve is given by:

$$\frac{1}{2}\int r^2 d\theta$$

Therefore:

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} \tan\theta \; d\theta$$

$$\implies A = \frac{1}{2}\left[\ln\sec\theta\right]_0^{\frac{\pi}{4}}$$

$$\implies A = \frac{1}{2}\left(\left(\ln\sec\frac{\pi}{4}\right) - (0)\right)$$

$$\implies A = \frac{1}{2}\ln\sqrt{2}$$

$$\implies A = \frac{1}{4}\ln 2$$

As the areas are the same we also need the area of B:

$$B = \int_{\frac{\pi}{4}}^{\beta} \tan\theta \; d\theta$$

We know that the area is the same so:

$$\frac{1}{4}\ln 2 = \int_{\frac{\pi}{4}}^{\beta} \tan\theta \; d\theta = \frac{1}{2}\left[\ln\sec\theta\right]_{\frac{\pi}{4}}^{\beta}$$

$$\implies \frac{1}{4}\ln 2 = \frac{1}{2}\left((\ln\sec\beta) - \left(\ln\sec\frac{\pi}{4}\right)\right)$$

$$\implies \frac{1}{4}\ln 2 = \frac{1}{2}\left((\ln\sec\beta) - \left(\ln\sqrt{2}\right)\right)$$

$$\implies \frac{1}{4}\ln 2 = \frac{1}{2}\left((\ln\sec\beta) - \left(\frac{1}{2}\ln 2\right)\right)$$

$$\implies \frac{1}{4}\ln 2 = \frac{1}{2}\ln\sec\beta - \frac{1}{4}\ln 2$$

$$\implies \frac{1}{2}\ln 2 = \frac{1}{2}\ln\sec\beta$$
$$\implies \ln 2 = \ln\sec\beta$$
$$\implies \sec\beta = 2$$
$$\implies \cos\beta = \frac{1}{2}$$
$$\implies \beta = \frac{\pi}{3}$$