Further Maths Core Pure 2 Practice Paper 2025

1. Let $f(x) = \cos x$. The mean value of $f(x)$ over the interval $[0, a]$ is 0.

(a) Write down a possible value of a.

(1)

Solution
A possible value is $a = \pi$, or any multiple of π.

(b) Let $g(x) = \arccos x$.

Find the exact mean value of $g(x)$ over the interval $[0, \frac{1}{2}]$.

(4)

Solution
The mean value is

$$\frac{1}{\frac{1}{2} - 0} \int_0^{\frac{1}{2}} \arccos(x)\, dx = \int_0^{\frac{1}{2}} 1 \times \arccos(x)\, dx$$

$$= 2\left([x \arccos x]_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\, dx \right)$$

using by parts

$$= 2\left([x \arccos x]_0^{\frac{1}{2}} - \frac{1}{2} \int_0^{\frac{1}{2}} \frac{-2x}{\sqrt{1-x^2}}\, dx \right)$$

$$= 2\left[x \arccos x - \sqrt{1-x^2} \right]_0^{\frac{1}{2}}$$

using reverse chain rule

$$= 2\left[\left(\frac{1}{2} \arccos \frac{1}{2} - \sqrt{1 - \frac{1}{4}} \right) - \left(0 - \sqrt{1} \right) \right]$$

$$= \arccos \frac{1}{2} - \sqrt{3} + 2$$

$$= \frac{\pi}{3} + 2 - \sqrt{3}$$

(Total for Question 1 is 5 marks)

2. (a) Prove by induction that

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$$

for all integers $n \geq 1$.

(5)

Solution

It's clearly true for $n = 1$ but you should write it out anyway.

Now assume it's true for $n = k$, i.e.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^k = \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix}$$

Now consider $n = k+1$:

$$\begin{aligned}
\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{k+1} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^k \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & \cos k\theta\sin\theta + \sin k\theta\cos\theta \\ -\sin k\theta\cos\theta - \sin\theta\cos k\theta & -\sin k\theta\sin\theta + \cos\theta\cos k\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos(k\theta+\theta) & \sin(k\theta+\theta) \\ -\sin(k\theta+\theta) & \cos(k\theta+\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta) \end{pmatrix}
\end{aligned}$$

Which is true for $n = k+1$.

True for $n = 1$. When true for $n = k$, it is also true for $n = k+1$. Therefore it is true for all integers n by induction.

(b) (i) Explain why $5^a - 15$ is always a multiple of 10 for all integer values of $a \geq 1$.

(1)

Solution

For all integers a, 5^a will end in a 5. Therefore when we subtract 15, the number will end in a 0. Therefore $5^a - 15$ is a multiple of 10.

(ii) Let $f(k) = 4^{2k} + 5^k - 1$.

By considering $f(k+1) - 16f(k)$, or otherwise, prove by induction that $f(k)$ is always divisible by 10 for all $k \geq 1$.

(4)

Solution

It's true for $n = 1$ as $f(1) = 60$ which is a multiple of 10.

Assume true for $n = k$, i.e. $f(k)$ is a multiple of 10.

$$\begin{aligned}
f(k+1) - 16f(k) &= 4^{2k+2} + 5^{k+1} - 1 - 16(4^2k + 5^k - 1) \\
&= 16 \times 4^{2k} + 5 \times 5^k - 1 - 16 \times 4^{2k} - 16 \times 5^k + 1
\end{aligned}$$

$$= 15 - 11 \times 5^k$$

As in part (b)(i) 5^k ends in a 5 and so does 11×5^k, therefore when we add 15 it must be a multiple of 10. Therefore $f(k+1)$ is divisible by 10.
True for $n = 1$. When true for $n = k$, it is also true for $n = k + 1$. Therefore it is true for all integers n by induction.

(Total for Question 2 is 10 marks)

3. The line l_1 has equation $\mathbf{r}_1 = \begin{pmatrix} 1 \\ a \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \\ b \end{pmatrix}$.

(a) The point $A(7, 2, -11)$ lies on the line l_1.

Show that $a = 6$ and $b = 5$.

(3)

Solution
The point lies on the line therefore

$$\begin{pmatrix} 1 \\ a \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \\ b \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ -11 \end{pmatrix}$$

The top line gives $1 - 3\lambda = 7 \implies \lambda = -2$.
The middle line gives $a + 2\lambda = 2$. As $\lambda = -2$, we get $a = 6$.
Similarly we find $b = 5$ by equating the last line.

(b) Find the shortest distance from the point $B(1, 2, -7)$ to the line l_1. Give your answer in the form $\sqrt{k}$, where k is a constant to be determined.

(4)

Solution
A general point on the line l_1 is given by

$$X = \begin{pmatrix} 1 - 3\lambda \\ 6 + 2\lambda \\ -1 + 5\lambda \end{pmatrix}$$

We want BX to be perpendicular to the line l_2 as the perpendicular distance is the shortest distance.

$$BX = \begin{pmatrix} -3\lambda \\ 4 + 2\lambda \\ 6 + 5\lambda \end{pmatrix}$$

The vectors are perpendicular if $BX \cdot \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix} = 0$.

Therefore we want

$$\begin{pmatrix} -3\lambda \\ 4 + 2\lambda \\ 6 + 5\lambda \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix} = 0$$

$$\implies 9\lambda + 8 + 4\lambda + 30 + 25\lambda = 0$$

$$38\lambda = -38$$

$$\lambda = -1$$

The shortest distance is given by finding $|BX|$.
Therefore the shortest distance is

$$\sqrt{(-3 \times -1)^2 + (4 + 2 \times -1}^2 + (6 + 5 \times -1)^2$$

$$= \sqrt{14}.$$

(Total for Question 3 is 7 marks)

4. The transformation represented by the 2×2 matrix A maps the point $(1, -3)$ to $(-9, 17)$ and has an invariant point at $(3, 1)$.

(a) Find the matrix A.

(3)

Solution

$(1, -3)$ maps to $(-9, 17)$ means

$$A\begin{pmatrix}1\\-3\end{pmatrix} = \begin{pmatrix}-9\\17\end{pmatrix}$$

Let $A = \begin{pmatrix}a & b\\c & d\end{pmatrix}$

$$\implies \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\-3\end{pmatrix} = \begin{pmatrix}-9\\17\end{pmatrix}$$

$$\implies a - 3b = -9, \text{ and } c - 3d = 17$$

Similarly as $(3, 1)$ is invariant:

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$$

Which gives

$$3a + b = 3, \text{ and } 3c + d = 1$$

Solving simultaneously gives us:

$$A = \begin{pmatrix}0 & 3\\2 & -5\end{pmatrix}$$

(b) Determine all invariant lines of A.

(5)

Solution

Consider a straight line $y = mx + c$.

An invariant line will satisfy:

$$A\begin{pmatrix}x\\mx+c\end{pmatrix} = \begin{pmatrix}X\\Y\end{pmatrix}$$

$$A\begin{pmatrix}x\\mx+c\end{pmatrix} = \begin{pmatrix}X\\Y\end{pmatrix}$$

$$\implies \begin{pmatrix}0 & 3\\2 & -5\end{pmatrix}\begin{pmatrix}x\\mx+c\end{pmatrix} = \begin{pmatrix}X\\Y\end{pmatrix}$$

$$\implies 3mx + 3c = X \text{ and } 2x - 5mx - 5c = Y$$

Therefore

$$x = \frac{X - 3c}{3m}$$

and

$$x = \frac{Y + 5c}{2 - 5m}$$

Which means

$$\frac{X - 3c}{3m} = \frac{Y + 5c}{2 - 5m}$$

$$\implies Y = \frac{2 - 5m}{3m}X - \frac{3c(2 - 5m)}{3m} - 5c$$

An invariant line satisfies $Y = mX + c$, so we have

$$\frac{2 - 5m}{3m} = m$$

$$\implies 3m^2 = 2 - 5m$$

$$\implies m = \frac{1}{3} \text{ or } m = -2$$

Now plugging in these values of m into the 'c' bit:
When $m = \frac{1}{3}$, the 'c' bit doesn't match up unless we have $c = 0$.
When $m = -2x + c$, we get 'c = c', therefore the invariant lines are

$$y = \frac{1}{3}x \text{ and } y = -2x + c.$$

(c) Find the image of the straight line $y = 2x + 1$ under the transformation A. Give your answer in the form $ay + bx + c = 0$, where a, b, and c are integers to be determined.

(4)

Solution

In a similar vain to the above:

$$A \begin{pmatrix} x \\ 2x + 1 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\implies \begin{pmatrix} 0 & 3 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x \\ 2x + 1 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\implies 6x + 3 = X \text{ and } 2x - 10x - 5 = Y$$

Therefore

$$x = \frac{X - 3}{6} \text{ and } x = \frac{Y + 5}{-8}$$

Therefore

$$\frac{X - 3}{6} = \frac{Y + 5}{-8}$$

$$\implies -8(X - 3) = 6(Y + 5)$$

$$\implies -8X + 24 = 6Y + 30$$

$$\implies 6Y + 8X + 6 = 0$$

$$\implies 3Y + 4X + 3 = 0$$

Therefore the image of the line is given by the equation $3y + 4x + 3 = 0$.

(Total for Question 4 is 12 marks)

5. You are given that

$$\sum_{k=3}^{18} f(k) = 728$$

for some function $f(k)$.

(a) Given that $f(k) = ak + b$, where a and b are constants, show that $21a + 2b = 91$.

(3)

Solution

$$\sum_{k=3}^{18} f(k) = 728$$

$$\implies \sum_{k=3}^{18} ak + b = 728$$

$$\implies a\sum_{k=3}^{18} k + b\sum_{k=3}^{18} 1 = 728$$

$$\implies a\left(\frac{1}{2}(18)(19) - \frac{1}{2}(2)(3)\right) + b(18-2) = 728$$

$$\implies a(171-3) + 16b = 728$$

$$\implies 168 + 16b = 728$$

$$\implies 84 + 8b = 364$$

$$\implies 21 + 2b = 91$$

(b) Given further that

$$\sum_{k=a}^{10} f(k) = 183$$

find the values of a and b.

(5)

Solution

$$\sum_{k=a}^{10} f(k) = 183$$

$$\implies \sum_{k=a}^{10} ak + b = 183$$

$$\implies a\sum_{k=a}^{10} k + b\sum_{k=a}^{10} 1 = 183$$

$$a\left(\frac{1}{2}(10)(11) - \frac{1}{2}(a-1)(a)\right) + b(10-(a-1)) = 183$$

$$a\left(55 - \frac{a^2}{2} + \frac{a}{2}\right) + 11b - ab = 183$$

$$\implies 55a - \frac{a^3}{2} + \frac{a^2}{2} + 11b - ab = 183$$
$$110a - a^3 + a^2 + 22b - 2ab = 366$$
$$110a - a^3 + a^2 + 11(91 - 2b) - a(91 - 2b) = 366$$

Using part (a)
After some more algebra you'll arrive at a cubic which has solution $a = 5$. Which then gives $b = -7$.

(Total for Question 5 is 8 marks)

6. (a) Prove that

$$\tan 5\theta \equiv \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$

(5)

Solution

Show that.

(b) Using part (a), or otherwise, prove that

$$\tan\frac{\pi}{5} = \sqrt{5 - 2\sqrt{5}}.$$

(4)

Solution

Show that.

(c) Prove also that

$$\sec^2\frac{\pi}{5} + \sec^2\frac{2\pi}{5} = 12.$$

(3)

Solution

Show that.

(Total for Question 6 is 12 marks)

7. The plane Π_1 has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and the plane Π_2 has equation $2x - 4y - z = 5$.

(a) Show that Π_1 and Π_2 are parallel planes.

(4)

Solution

(b) Find the vertical distance between the planes Π_1 and Π_2.

(4)

Solution

(c) A third plane Π_3 has equation $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 4$.

Find the acute angle between Π_1 and Π_3.

(3)

Solution

83.7 degrees

(Total for Question 7 is 11 marks)

8. The polar curve C_1 has equation $r = 2\sec\left(\theta + \frac{\pi}{3}\right)$.

(a) Prove that C_1 is a straight line and find its equation.

(3)

Solution
$\sqrt{3}y - x + 4 = 0$.

(b) The polar curve C_2 has equation $r = 3 + \cos^2\theta$.

(i) Show that the Cartesian equation of C_2 is $(x^2 + y^2)^{\frac{3}{2}} = 4x^2 + 3y^2$.

(3)

Solution
Show that.

(ii) Prove that the gradient function of C_2 can be written in the form

$$\frac{dy}{dx} = \frac{-\cot\theta(a\cos^2\theta + 1)}{b(\cos^2\theta + 1)}$$

where a and b are integers to be determined. **(4)**

Solution
$a = b = 3$

(Total for Question 8 is 10 marks)